MATH 105A and 110A Review: Partial derivatives, finding the gradient and Jacobian matrix

1. Find the gradient of $f(x, y, z) = x^2y + y$.

Solution: Note that $f : \mathbb{R}^3 \to \mathbb{R}$. We need $f_x = 2xy, f_y = x^2 + 1, f_z = 0$. Thus,

$$\nabla f = (2xy, x^2 + 1, 0).$$

2. Find the Jacobian matrix of $F(x, y, z) = (xyz, x + z, y)$ at the point $(1, 2, 2)$.

Solution: Since $F : \mathbb{R}^3 \to \mathbb{R}^3$, the Jacobian matrix is a 3×3 matrix. Let

$$\begin{aligned} f_1(x, y, z) &= xyz, \\ f_2(x, y, z) &= x + z, \\ f_3(x, y, z) &= y. \end{aligned}$$

Then,

$$\begin{aligned} \nabla f_1 &= (yz, xz, xy), \\ \nabla f_2 &= (1, 0, 1), \\ \nabla f_3 &= (0, 1, 0). \end{aligned}$$

Hence,

$$\nabla F = \begin{bmatrix} yz & xz & xy \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

At the point $(1, 2, 2)$, the Jacobian matrix is

$$\nabla F = \begin{bmatrix} 4 & 2 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

3. Find the Jacobian matrix of $F(x, y, z) = (xz, zy + x)$.

Solution: Since $F : \mathbb{R}^3 \to \mathbb{R}^2$, the Jacobian matrix is a 2×3 matrix. Let

$$\begin{aligned} f_1(x, y, z) &= xz, \\ f_2(x, y, z) &= zy + x. \end{aligned}$$

Then,

$$\begin{aligned} \nabla f_1 &= (z, 0, x), \\ \nabla f_2 &= (1, z, y). \end{aligned}$$

Hence,

$$\nabla F = \begin{bmatrix} z & 0 & x \\ 1 & z & y \end{bmatrix}.$$